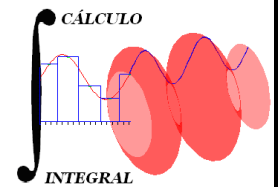




UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
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DIVISIÓN DE CIENCIAS BÁSICAS
COORDINACIÓN DE MATEMÁTICAS
CÁLCULO INTEGRAL
SEGUNDO EXAMEN FINAL COLEGIADO



3 de junio de 2019

Semestre 2019-2

INSTRUCCIONES: Lee cuidadosamente los enunciados de los **6 reactivos** que componen el examen antes de empezar a resolverlos. La duración máxima del examen es de **2 horas**.

1. Obtén la serie de Taylor de la función

$$f(x) = e^{2x}$$

alrededor de $a = 1$

15 puntos

2. Determina si la siguiente integral converge o diverge

$$\int_{-1}^3 \frac{dx}{\sqrt{1+x}}$$

15 puntos

3. Efectúa:

a) $\int \frac{dx}{x^2(x-1)}$

b) $\int \frac{x}{\sqrt{1+x^4}} dx$

c) $\int \operatorname{ang} \tan x dx$

30 puntos

4. Calcula la longitud de arco de la gráfica de la función $f(x) = 2\sqrt{x^3}$ en el intervalo $\left[0, \frac{1}{3}\right]$.

10 puntos

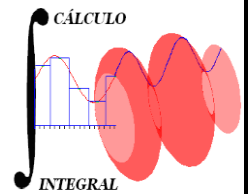
5. Sean $f(u, v) = u^2 - v$ y $u = e^{2x-y}$, $v = \ln(xy)$, calcula

$$\left. \frac{\partial f}{\partial y} \right|_{(1, 1)}$$

15 puntos

6. Obtén la ecuación cartesiana del plano tangente a la gráfica de la función $z = -x^2 - y^2 + 4$ en el punto $P(1, 1)$

15 puntos



1. Sea:

$$f(x) = e^{2x} \quad ; \quad a = 1$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \dots$$

$$f(x) = e^{2x} \qquad f(1) = e^2$$

$$f'(x) = 2e^{2x} \qquad f'(1) = 2e^2$$

$$f''(x) = 4e^{2x} \qquad f''(1) = 4e^2$$

$$f'''(x) = 8e^{2x} \qquad f'''(1) = 8e^2$$

⋮

$$e^{2x} = e^2 + 2e^2(x-1) + \frac{1}{2!} 4e^2(x-1)^2 + \frac{1}{3!} 8e^2(x-1)^3 + \dots$$

$$e^{2x} = e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4}{3}e^2(x-1)^3 + \dots$$

15 puntos

2. Es impropia:

$$\int_{-1}^3 \frac{dx}{\sqrt{1+x}} \quad 1+x > 0; x > -1 \quad \text{No continua en } x = -1$$

$$\int_{-1}^3 \frac{dx}{\sqrt{1+x}} = \lim_{\varepsilon \rightarrow 0^+} \left(\int_{-1+\varepsilon}^3 \frac{dx}{\sqrt{1+x}} \right)$$

$$\text{Si } I = \int \frac{dx}{\sqrt{1+x}} \quad \left| \begin{array}{l} u = 1+x \\ du = dx \end{array} \right.$$

$$I = \int \frac{du}{u^{1/2}} = \int u^{-1/2} du = \frac{u^{1/2}}{1/2} + C = 2u^{1/2} + C$$

$$I = 2\sqrt{1+x} + C$$

$$\Rightarrow \int_{-1}^3 \frac{dx}{\sqrt{1+x}} = \lim_{\varepsilon \rightarrow 0^+} \left(\left[2\sqrt{1+x} \right]_{-1+\varepsilon}^3 \right)$$

$$= \lim_{\varepsilon \rightarrow 0^+} 2 \left(\sqrt{1+3} - \sqrt{1-1+\varepsilon} \right)$$

$$= \lim_{\varepsilon \rightarrow 0^+} 2 \left(2 - \sqrt{\varepsilon} \right) = 4$$

$$\int_{-1}^3 \frac{dx}{\sqrt{1+x}} = 4 \quad \therefore \text{Convergente}$$

3. Solución

a) Por descomposición en fracciones parciales

$$\text{Sea } \frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1};$$

$$1 = Ax(x-1) + B(x-1) + Cx^2$$

$$1 = A(x^2 - x) + B(x-1) + Cx^2$$

$$A + C = 0 \quad \boxed{A = -1}$$

$$-A + B = 0 \quad \boxed{B = -1}$$

$$-B = 1 \quad \boxed{C = 1}$$

$$\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$$

$$I = \int \left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1} \right) dx = -\int \frac{dx}{x} - \int \frac{dx}{x^2} + \int \frac{dx}{x-1}$$

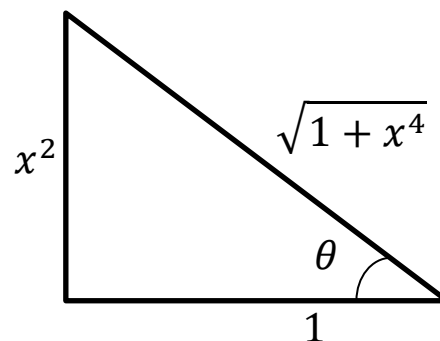
$$I = -\ln(x) + \frac{1}{x} + \ln(x-1) + C \Rightarrow \boxed{I = \ln\left(\frac{x-1}{x}\right) + \frac{1}{x} + C}$$

b) Por sustitución trigonométrica

$$x^2 = \tan(\theta)$$

$$2x dx = \sec^2(\theta) d\theta; \quad x dx = \frac{1}{2} \sec^2(\theta) d\theta$$

$$\sqrt{1+x^4} = \sec(\theta)$$



Al sustituir en la integral:

$$I = \frac{1}{2} \int \frac{\sec^2(\theta) d\theta}{\sec(\theta)} = \frac{1}{2} \int \sec(\theta) d\theta$$

$$I = \frac{1}{2} \ln(\tan(\theta) + \sec(\theta)) + C \Rightarrow \boxed{I = \ln\left(\sqrt{1+x^4} + x^2\right)^{\frac{1}{2}} + C}$$

c) Por partes

$$\left| \begin{array}{l} u = \text{ang tan}(x) \\ du = \frac{dx}{1+x^2} \end{array} \right. \Rightarrow \left| \begin{array}{l} dv = dx \\ v = x \end{array} \right.$$

$$I = x \text{ang tan}(x) - \int \frac{x dx}{1+x^2}$$

$$\omega = 1 + x^2$$

$$d\omega = 2x dx$$

$$x dx = \frac{1}{2} d\omega$$

$$\int \frac{x dx}{1+x^2} = \frac{1}{2} \int \frac{d\omega}{\omega} = \frac{1}{2} \ln(\omega) + C$$

$$\int \frac{x dx}{1+x^2} = \frac{1}{2} \ln(1+x^2) + C$$

$$I = x \text{ang tan}(x) - \frac{1}{2} \ln(1+x^2) + C$$

4.

$$f(x) = 2\sqrt{x^3} = 2x^{3/2} \quad ; \quad \left[0, \frac{1}{3}\right]$$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$f'(x) = 2 \cdot \frac{3}{2} x^{1/2} = 3x^{1/2} \quad \therefore \quad [f'(x)]^2 = 9x$$

$$L = \int_0^{1/3} \sqrt{1+9x} dx$$

$$u = 1 + 9x$$

$$du = 9dx$$

$$dx = \frac{1}{9} du$$

$$\int \sqrt{1+9x} dx = \frac{1}{9} \int u^{1/2} du = \frac{1}{9} \left[\frac{u^{3/2}}{3/2} \right] + C$$

$$L = \frac{2}{27} \left[(1+9x)^{3/2} \right]_0^{1/3} = \frac{2}{27} \left[\left(1 + 9 \left(\frac{1}{3} \right) \right)^{3/2} - 1 \right]$$

$$L = \frac{2}{27} [7]$$

$$\boxed{L = \frac{14}{27} u}$$

10 Puntos

5.

$$f = u^2 - v \quad ; \quad u = e^{2x-y} \quad ; \quad v = \ln(xy)$$

$$\left. \frac{\partial f}{\partial y} \right|_{(1,1)} = ?$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial u} = 2u; \quad \frac{\partial f}{\partial v} = -1; \quad \frac{\partial u}{\partial y} = -e^{2x-y}; \quad \frac{\partial v}{\partial y} = \frac{x}{xy} = \frac{1}{y}$$

$$\frac{\partial f}{\partial y} = 2u \cdot (-e^{2x-y}) + (-1) \cdot \frac{1}{y}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{y} - 2u(e^{2x-y}) = -\frac{1}{y} - 2(e^{2x-y})(e^{2x-y})$$

$$\frac{\partial f}{\partial y} = -\frac{1}{y} - 2(e^{2x-y})^2$$

$$\left. \frac{\partial f}{\partial y} \right|_{(1,1)} = -\frac{1}{1} - 2(e^{2(1)-1})^2 = \boxed{-1 - 2e^2}$$

6.

$$z = -x^2 - y^2 + 4; \quad P(1,1,2)$$

$$-(x - x_0) \frac{\partial z}{\partial x} - (y - y_0) \frac{\partial z}{\partial y} + (z - z_0) = 0$$

$$\frac{\partial z}{\partial x} = -2x; \quad \left. \frac{\partial z}{\partial x} \right|_P = -2$$

$$\frac{\partial z}{\partial y} = -2y; \quad \left. \frac{\partial z}{\partial y} \right|_P = -2$$

$$-(x-1)(-2) - (y-1)(-2) + (z-2) = 0$$

$$2x - 2 + 2y - 2 + z - 2 = 0$$

$$\Rightarrow \boxed{2x + 2y + z - 6 = 0}$$

15 Puntos