Integración de Funciones Polinomiales

Ecuación Diferencial		Solución Particular		Solución
1	$y' - y = 8$	$y_P = \frac{1}{D-1}(8)$		$y_P = \frac{1}{-1}(8) = -8$
2	$y' - y = x$	$y_P = \frac{1}{D-1}(x)$	$-1 - D - \ldots$ $-1 + D \overline{)\ 1}$ $-1 + D$	$y_P = (-1 - D)x = -x - 1$
3	$y' - y = x^3$	$y_P = \frac{1}{D-1}\left(x^3\right)$	$-1 - D - D^2 - D^3 - \ldots$ $-1 + D \overline{)\ 1}$ $-1 + D$ $-D + D^2$ $-D^2 + D^3$	$y_P = \left(-1 - D - D^2 - D^3\right)x^3$ $y_P = -x^3 - 3x^2 - 6x - 6$
4	$y'' - 3y' + 2y = 4$	$y_P = \frac{1}{D^2 - 3D + 2}(4)$		$y_P = \frac{4}{2} = 2$
5	$y'' - 3y' + 2y = x$	$y_P = \frac{1}{D^2 - 3D + 2}(x)$	$\frac{1}{2} + \frac{3D}{4} + \ldots$ $2 - 3D \overline{)\ 1}$ $-1 + \frac{3D}{2}$	$y_P = \left(\frac{1}{2} + \frac{3D}{4}\right)x = \frac{1}{2}x + \frac{3}{4}$
6	$y''' + y'' - 3y' + 2y = x + x^2$	$y_P = \frac{1}{D^3 + D^2 - 3D + 2}\left(x + x^2\right)$	$\frac{1}{2} + \frac{3D}{4} + \frac{7D^2}{8}$ $2 - 3D + D^2 \overline{)1}$ $-1 + \frac{3D}{2} - \frac{D^2}{2}$ $-\frac{3D}{2} + \frac{9D^2}{4}$ $\frac{7D^2}{4}$	$y_P : \left(\frac{1}{2} + \frac{3D}{4} + \frac{7D^2}{8}\right)x^2$ $= \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}$ $\left(\frac{1}{2} + \frac{3D}{4} + \frac{7D^2}{8}\right)x = \frac{1}{2}x + \frac{3}{4}$ $y_P = \frac{1}{2}x^2 + 2x + \frac{5}{2}$

Primera Propiedad del Operador Inverso

a) Si $y_P = \frac{1}{P(D)} e^{+ax}$ y no hay raíces repetidas, entonces $y_P = \frac{1}{P(+a)} e^{+ax}$

$$y_P = \frac{1}{D^2 - 3D + 2} e^{+4x} = \frac{1}{(D-1)(D-2)} e^{+4x} = \frac{1}{(4-1)(4-2)} e^{+4x} = \frac{1}{6} e^{+4x}$$

b) Si $y_P = \frac{1}{P(D)} e^{-bx}$ y no hay raíces repetidas, entonces $y_P = \frac{1}{P(-b)} e^{-bx}$

$y_P = \frac{1}{D^2 - 3D + 2} e^{-3x}$ como no hay raíces repetidas, entonces $y_P = \frac{1}{(-3)^2 - 3(-3) + 2} e^{-3x} = \frac{1}{20} e^{-3x}$

Segunda Propiedad del Operador D

Si $y_P = \frac{1}{P(D)} sen\, ax$ donde $a \in \mathbb{R}$ y no hay raíces repetidas entonces $y_P = \frac{1}{P(D)} \bigg|_{D^2 por -a^2}^{sen\, ax}$

$$y_P = \frac{1}{D^2 - 3D + 2} sen2x = \frac{1}{-3D - 2} sen2x = \frac{-1}{3D + 2} sen2x = \frac{-1(3D-2)}{9D^2 - 4} sen\, 2x$$

$$y_P = \frac{-(3D-2)}{-40} sen2x = \frac{1}{40}(3D-2) sen2x = \frac{6}{40}\cos 2x - \frac{2}{40} sen2x$$

$$y_P = \frac{3}{20}\cos 2x - \frac{1}{20} sen2x$$

$$y_P = \frac{1}{D^2 - 3D + 2}\cos 3x = \frac{+1}{-3D - 7}\cos 3x = \frac{-1}{3D + 7}\cos 3x = \frac{-(3D-7)}{9D^2 - 49}\cos 3x$$

$$y_P = \frac{1}{130}(3D-7)\cos 3x = -\frac{9}{130} sen3x - \frac{7}{130}\cos 3x$$

Tercera Propiedad del Operador Inverso

a) Si $y_P = \frac{1}{P(D)} e^{+ax} y(x) = e^{+ax} \frac{1}{P(D)} \bigg|_{D \to D+a} y(x) = e^{+ax} \frac{1}{P(D-a)} y(x)$

b) Si $y_P = \frac{1}{P(D)} e^{-ax} y(x) = e^{-ax} \frac{1}{P(D)} \bigg|_{D \to D-a} y(x) = e^{-ax} \frac{1}{P(D-a)} y(x)$

1) $y_P = \frac{1}{D^2 - 3D + 2}\left(xe^x\right) = \frac{1}{(D-1)(D-2)} xe^x = e^x \frac{1}{D}\left[\frac{1}{D-1} x\right] = \left(-\frac{1}{2}x^2 - x\right)e^x$

2) $y_P = \frac{1}{D^2 - 3D + 2} e^{2x} sen2x = e^{2x} \frac{1}{D(D+1)} sen2x = e^{2x} \frac{1}{D^2 + D} sen2x = e^{2x} \frac{1}{D-4} sen2x$

$= e^{2x} \frac{D+4}{D^2 - 16} sen2x = -\frac{1}{20} e^{2x}(D+4) sen2x = -\frac{1}{10} e^{2x} \cos 2x - \frac{1}{5} e^{2x} sen2x$

Cuarta Propiedad del Operador Derivada

Si $y_P = \frac{1}{P(D)} x^n y(x) = \left[x + \frac{d}{dD}\right]^n \frac{1}{P(D)} y(x)$

Esta propiedad <u>no es aplicable</u> cuando hay raíces repetidas

$$y_P = \frac{1}{D^2 - 3D + 2} x^2 e^{3x}$$

$$y_P = \left(x + \frac{d}{dD}\right)^2 \frac{1}{D^2 - 3D + 2} e^{3x} = \left(x^2 + 2x\frac{d}{dD} + \frac{d^2}{dD}\right) \frac{1}{D^2 - 3D + 2} e^{3x}$$

$$= x^2 \frac{1}{D^2 - 3D + 2} e^{3x} = \frac{1}{2} x^2 e^{3x}$$

$$+2x\left(\frac{d}{dD} \frac{1}{D^2 - 3D + 2}\right) e^{3x} = -2x \frac{2D-3}{\left(D^2 - 3D + 2\right)^2} e^{3x} = -\frac{3}{2} xe^{3x}$$

$$+\frac{d^2}{dD^2}\left(\frac{1}{D^2-3D+2}\right)e^{3x}=\frac{-2\left(D^2-3D+2\right)+2\left(2D-3\right)^2}{\left(D^2-3D+2\right)^3}e^{3x}=\frac{7}{4}e^{3x}$$

Mediante la tercera propiedad:

$$y_P=\frac{1}{(D-1)(D-2)}x^2e^{3x}=e^{3x}\frac{1}{(D+2)(D+1)}x^2=\frac{1}{2}x^2e^{3x}-\frac{3}{2}xe^{3x}+\frac{7}{4}e^{3x}$$

Ecuaciones Diferenciales cuyos términos independientes son funciones senoidales o cosenoidales

Emplearemos la siguiente propiedad:

$$\int x^n e^{\beta x_i}dx=Términos\ reales\ +\ i\ Términos\ reales$$

$$\int x^n \cos\beta x\,dx=Términos\ reales$$

$$\int x^n sen\beta x\,dx=Términos\ reales$$

Si $y_P=\frac{1}{D^2+\beta^2}(x^0,x,x^2,...,x^n)sen\beta x\ o\ \cos\beta x$

Sustituyendo $sen\beta x$ o $\cos\beta x$ por $e^{\beta xi}$, se tiene:

$$\bar{y}_P=\frac{1}{D^2+\beta^2}e^{\beta ix}$$

Usando la tercera propiedad

$$\bar{y}_P=e^{\beta xi}\frac{1}{(D+\beta i)^2+\beta^2}(\mathrm{x}^0,\mathrm{x},...\mathrm{x}^n)=\mathrm{e}^{\beta xi}\frac{1}{D}\left[\frac{1}{D+2\beta i}(\mathrm{x}^0,\mathrm{x},...,\mathrm{x}^n)\right]$$

$$\bar{y}_P=e^{\beta xi}\frac{1}{D}\left(\frac{2}{2\beta i}-\frac{D}{(2\beta i)^2}+\frac{D^2}{(2\beta i)^3}-...\right)(\mathrm{x}^0,\mathrm{x},\mathrm{x}^2,...,\mathrm{x}^n)$$

Finalmente

$$\bar{y}_P=\left(\sum_{R=1}^{n+1}\frac{-(i)^k D^{(k-2)}}{(2\beta i)^k}x^n\right)\left(\cos\beta x+i\,sen\beta x\right)$$

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Resolver la Ecuación Diferencial

$$y'' - 4y' + 13y = x^2 e^{2x} sen3x$$

$$\left(D^2 - 4D + 13\right) y =$$

$$\left[\left(D-2\right)^2 + 9\right] y = x^2 e^{2x} sen3x$$

$$y_H = C_1 e^{2x} \cos 3x + C_2 e^{2x} sen3x$$

$$y_P = \frac{1}{\left(D-2\right)^2 + 9} x^2 e^{2x} sen3x$$ raíces repetidas por la tercera propiedad

$$y_P = e^{2x} \frac{1}{D^2 + 9} x^2 sen3x \begin{cases} x^n = x^2 \\ n = 2 \qquad n+1 = 3 \\ \beta = 3 \qquad 2\beta = 6 \end{cases}$$ términos de la serie

$$\overline{y}_P = \left(\sum_{k=1}^{3} \frac{-(i)^k D^{k-2}}{(2\beta)^k} x^n \right) \left(\cos \beta x + i sen \beta x\right)$$

$$= \left[\frac{-iD^{-1}}{6} x^2 + \frac{1}{36} x^2 + \frac{i}{196} Dx^2 \right] \left(\cos 3x + i sen3x\right)$$

$$\overline{y}_P = \left[-\frac{i}{18} x^3 + \frac{1}{36} x^2 + \frac{ix}{98} \right] \left(\cos 3x + i sen3x\right)$$

$$y_P = -\frac{1}{18} x^3 e^{2x} \cos 3x + \frac{1}{36} x^2 e^{2x} sen3x + \frac{1}{98} e^{2x} \cos 3x$$

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